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Hawking Radiation from a General Spherically Symmetric Evaporating Black Hole

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By means of generalized tortoise coordinates both the Klein–Gordon equation and the Dirac equation are reduced near the event horizon of a general spherically symmetric evaporating black hole. The location and the temperature of the event horizon are given automatically without calculating the energy-momentum tensor. The Hawking thermal spectra of the Klein–Gordon particles and the Dirac particles are obtained, respectively.

Recently we have proposed a new method for determining the Hawking effect of a general spherically symmetric evaporating black hole without calculating the vacuum expectation value of the renormalized energy-momentum tensor (Zhao and Dai, 1991, 1992). In this paper, the method is improved, and not only the Klein–Gordon equation, but also the Dirac equation are studied.

The line element of the space-time where there exists a general spherically symmetric evaporating black hole is given as (Balbinot and Barletta, 1989)

$$ds^2 = -e^{2\psi}\left(1-\frac{2m}{r}\right)dv^2 + 2e^{\psi}\,dv\,dr + r^2\,d\Omega^2 \tag{1}$$

where $m = m(r, v)$, $\psi = \psi(r, v)$. After the separation of variables

$$\Phi_{\omega lm} = \frac{1}{r}\,\rho(r, v)\,Y_{lm}(\theta, \phi) \tag{2}$$

the radial part of the Klein–Gordon equation is given as

$$\left(1-\frac{2m}{r}\right)\frac{\partial^2\rho}{\partial r^2} + 2e^{-\psi}\frac{\partial^2\rho}{\partial r\,\partial v} + A\frac{\partial\rho}{\partial r} - B\rho = 0 \tag{3}$$

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where

$$A \equiv \left(1 - \frac{2m}{r}\right)\frac{\partial\psi}{\partial r} - \frac{2(m'r - m)}{r^2}$$

$$B \equiv \frac{A}{r} + \mu_0^2 + \frac{l(l+1)}{r^2} \tag{4}$$

$$m' \equiv \frac{\partial m}{\partial r}, \qquad \psi' \equiv \frac{\partial \psi}{\partial r}$$

Introducing the generalized tortoise coordinates

$$r_* = r + \frac{1}{2\varkappa}\ln[r - r_{\rm H}(v)]$$
$$v_* = v - v_0 \tag{5}$$

we can write equation (3) as

$$\frac{[2\varkappa(r - r_{\rm H}) + 1](r - 2m)e^{\psi} - 2r\dot{r}_{\rm H}}{2\varkappa r(r - r_{\rm H})}\frac{\partial^2 \rho}{\partial r_*^2} + 2\frac{\partial^2 \rho}{\partial r_*\,\partial v_*}$$
$$+ \left\{\frac{2r\dot{r}_{\rm H} - (r - 2m)e^{\psi}}{r(r - r_{\rm H})[2\varkappa(r - r_{\rm H}) + 1]} + Ae^{\psi}\right\}\frac{\partial \rho}{\partial r_*}$$
$$- \frac{2\varkappa(r - r_{\rm H})e^{\psi}B}{2\varkappa(r - r_{\rm H}) + 1}\rho = 0 \tag{6}$$

where $r_{\rm H}$ is the event horizon. $\varkappa$ is an adjustable temperature parameter. It is constant under the tortoise transformation (5). $r_{\rm H}$ can be given by the null-surface condition

$$g^{\mu\nu}\frac{\partial f}{\partial x^\mu}\frac{\partial f}{\partial x^\nu} = 0 \tag{7}$$

Substituting the metric (1) into equation (7), we have

$$(r - 2m)e^{\psi} - 2r\frac{dr}{dv} = 0 \tag{8}$$

Evidently, $r_{\rm H}(v)$ is the solution of the equation. This means

$$r_{\rm H} = 2M/(1 - 2\dot{r}_{\rm H}e^{-\psi}) \tag{9}$$

where

$$\dot{r}_{\rm H} \equiv \frac{dr_{\rm H}}{dv}, \qquad M \equiv m(r = r_{\rm H}, v)$$

When r goes to $r_H(v_0)$ and v goes to v_0, using equation (9), we can reduce equation (6) to

$$A\frac{\partial^2\rho}{\partial r_*^2}+2\frac{\partial^2\rho}{\partial r_*\,\partial v_*}=0 \tag{10}$$

where

$$A=\lim_{\substack{r\to r_H(v_0)\\ v\to v_0}}\frac{[2\varkappa(r-r_H)+1](r-2m)e^{\psi}-2r\dot{r}_H}{2\varkappa r(r-r_H)} \tag{11}$$

Selecting the adjustable parameter $\varkappa$ as

$$\varkappa=\left.\frac{1-2m'-2\dot{r}_He^{-\psi}(1-r_H\psi')}{4M+2r_H(e^{-\psi}-1)}\right|_{v=v_0} \tag{12}$$

we have $A=1$, and equation (10) can be reduced to

$$\frac{\partial^2\rho}{\partial r_*^2}+2\frac{\partial^2\rho}{\partial r_*\,\partial v_*}=0 \tag{13}$$

Its ingoing wave solution and outgoing wave solution are, respectively,

$$\rho_{\text{in}}=e^{-i\omega v_*} \tag{14}$$

$$\rho_{\text{out}}=e^{-i\omega v_*+2i\omega r_*} \tag{15}$$

Equation (15) can be rewritten as

$$\rho_{\text{out}}=e^{-i\omega v_*+2i\omega r}(r-r_H)^{i\omega/\varkappa} \tag{16}$$

It is not analytical at the horizon $r=r_H$. We extend it by analytical continuation to the inside of the black hole through the lower half complex r-plane

$$(r-r_H)\to|r-r_H|e^{-i\pi}=(r_H-r)e^{-i\pi} \tag{17}$$

$$\rho_{\text{out}}\to\rho'_{\text{out}}=e^{-i\omega v_*+2i\omega r}(r_H-r)^{i\omega/\varkappa}\,e^{\pi\omega/\varkappa} \tag{18}$$

The scattering probability of the outgoing wave at the horizon is

$$\left|\frac{\rho_{\text{out}}}{\rho'_{\text{out}}}\right|^2=e^{-2\pi\omega/\varkappa} \tag{19}$$

Following Damour and Ruffini (1976) and Sannan (1988), it is easy to obtain the spectrum of radiation of the Klein–Gordon particles from the black hole

$$N_\omega = [\exp(\omega/k_B T) - 1]^{-1} \tag{20}$$

$$T = \frac{\varkappa}{2\pi k_B} = \frac{1}{2\pi k_B} \frac{1 - 2m' - 2\dot{r}_H e^{-\psi}(1 - r_H \psi')}{4M + 2r_H(e^{-\psi} - 1)} \tag{21}$$

where k_B is Boltzmann's constant.

Now let us study the Hawking radiation of the Dirac particles from the black hole.

With the signature -2, the spinor base form of the Dirac equation in curved space-time is (Page, 1976)

$$\begin{aligned} \sqrt{2}\,\nabla_{a\dot{b}} P^a + i\mu_0 \bar{Q}_{\dot{b}} = 0 \\ \sqrt{2}\,\nabla_{a\dot{b}} Q^a + i\mu_0 \bar{P}_{\dot{b}} = 0 \end{aligned} \tag{22}$$

where μ_0 is the mass of the Dirac particle. P^a, Q^a, and $\nabla_{a\dot{b}}$ are, respectively, the 2-component spinors and the covariant spinor differentiation expressed with spinor base components. It can be transformed into four coupled equations,

$$\begin{aligned} (D + \varepsilon - \rho)F_1 + (\bar{\delta} + \pi - \alpha)F_2 &= i\mu_0 G_1/\sqrt{2} \\ (\Delta + \mu - \gamma)F_2 + (\delta + \beta - \tau)F_1 &= i\mu_0 G_2/\sqrt{2} \\ (D + \bar{\varepsilon} - \bar{\rho})G_2 - (\delta + \bar{\pi} - \bar{\alpha})G_1 &= i\mu_0 F_2/\sqrt{2} \\ (\Delta + \bar{\mu} - \bar{\gamma})G_1 - (\bar{\delta} + \bar{\beta} - \bar{\tau})G_2 &= i\mu_0 F_1/\sqrt{2} \end{aligned} \tag{23}$$

where

$$\begin{aligned} F_1 = P^0, \qquad F_2 = P^1, \qquad G_1 = \bar{Q}^{\dot{1}}, \qquad G_2 = -\bar{Q}^{\dot{0}} \\ D = \partial_{0\dot{0}} = l^\mu \partial_\mu, \qquad \Delta = \partial_{1\dot{1}} = n^\mu \partial_\mu \\ \delta = \partial_{0\dot{1}} = m^\mu \partial_\mu, \qquad \bar{\delta} = \bar{m}^\mu \partial_\mu \end{aligned} \tag{24}$$

μ, γ, β, τ, ε, ρ, π, and α are the special designations of the spin coefficients defined by Newman and Penrose (1962). l^μ, n^μ, m^μ, and $\bar{m}^\mu$ are the null tetrad vectors,

$$\begin{aligned} l_\mu l^\mu = n_\mu n^\mu = m_\mu m^\mu = \bar{m}_\mu \bar{m}^\mu = 0 \\ l_\mu m^\mu = l_\mu \bar{m}^\mu = n_\mu m^\mu = n_\mu \bar{m}^\mu = 0 \\ l_\mu n^\mu = -m_\mu \bar{m}^\mu = 1 \\ g_{\mu\nu} = l_\mu n_\nu + n_\mu l_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu \end{aligned} \tag{25}$$

In the general spherically symmetric space-time (1), they are

$$\begin{aligned} l_\mu &= (-e^\psi, 0, 0, 0) \\ n_\mu &= [-(1/2)\, e^\psi(1-2m/r), 1, 0, 0] \\ m_\mu &= (r/\sqrt{2})(0, 0, 1, -i\sin\theta) \\ \bar{m}_\mu &= (r/\sqrt{2})(0, 0, 1, i\sin\theta) \end{aligned} \tag{26}$$

and

$$\begin{aligned} l^\mu &= (0, 1, 0, 0) \\ n^\mu &= [-e^{-\psi}, -(1/2)(1-2m/r), 0, 0] \\ m^\mu &= (1/\sqrt{2}\, r)(0, 0, -1, i/\sin\theta) \\ \bar{m}^\mu &= (1/\sqrt{2}\, r)(0, 0, -1, -i/\sin\theta) \end{aligned} \tag{27}$$

Then, equation (23) can be reduced to

$$\begin{aligned} &\left(\frac{\partial}{\partial r}+\frac{1}{2}\psi'+\frac{1}{r}\right)F_1-\frac{1}{\sqrt{2}\,r}\left(\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\partial}{\partial\phi}+\frac{1}{2}\operatorname{ctg}\theta\right)F_2=i\frac{\mu_0}{\sqrt{2}}G_1 \\ &\left(e^{-\psi}\frac{\partial}{\partial v}+\frac{1}{2}\left(1-\frac{2m}{r}\right)\frac{\partial}{\partial r}+\frac{1}{2r}\left(1-\frac{m}{r}\right)+\frac{1}{4}\left(1-\frac{2m}{r}\right)\psi'-\frac{m'}{2r}\right)F_2 \\ &\qquad+\frac{1}{\sqrt{2}\,r}\left(\frac{\partial}{\partial\theta}-\frac{i}{\sin\theta}\frac{\partial}{\partial\phi}+\frac{1}{2}\operatorname{ctg}\theta\right)F_1=-i\frac{\mu_0}{\sqrt{2}}G_2 \\ &\frac{1}{\sqrt{2}\,r}\left(\frac{\partial}{\partial\theta}-\frac{i}{\sin\theta}\frac{\partial}{\partial\phi}+\frac{1}{2}\operatorname{ctg}\theta\right)G_1+\left(\frac{\partial}{\partial r}+\frac{1}{2}\psi'+\frac{1}{r}\right)G_2=i\frac{\mu_0}{\sqrt{2}}F_2 \\ &\frac{1}{\sqrt{2}\,r}\left(\frac{\partial}{\partial\theta}-\frac{i}{\sin\theta}\frac{\partial}{\partial\phi}+\frac{1}{2}\operatorname{ctg}\theta\right)G_2-\left(e^{-\psi}\frac{\partial}{\partial v}+\frac{1}{2}\left(1-\frac{2m}{r}\right)\frac{\partial}{\partial r}\right. \\ &\qquad\left.+\frac{1}{2r}\left(1-\frac{m}{r}\right)\psi'-\frac{m'}{2r}\right)G_1=i\frac{\mu_0}{\sqrt{2}}F_1 \end{aligned} \tag{28}$$

After the separation of variables

$$\begin{aligned} F_1 &= e^{iv\phi}R_-(v,r)\,S_-(\theta) \\ F_2 &= e^{iv\phi}R_+(v,r)\,S_+(\theta) \\ G_1 &= e^{iv\phi}R_+(v,r)\,S_-(\theta) \\ G_2 &= e^{iv\phi}R_-(v,r)\,S_+(\theta) \end{aligned} \tag{29}$$

the radial part of the decoupled Dirac equation about R_+ can be written as

$$e^{\psi}\left(1-\frac{2m}{r}\right)\frac{\partial^2 R_+}{\partial r^2}+2\frac{\partial^2 R_+}{\partial r\,\partial v}+\left(\frac{\psi}{r}-\psi'+\frac{i2\mu_0}{\lambda-i\mu_0 r}\right)\frac{\partial R_+}{\partial v}$$
$$+e^{\psi}\left[\left(1-\frac{2m}{r}\right)\psi'-\frac{3m'}{r}+\frac{3}{r^2}(r-m)\right.$$
$$\left.+\frac{i\mu_0}{\lambda-i\mu_0 r}\left(1-\frac{2m}{r}\right)\right]\frac{\partial R_+}{\partial r}+B_+R_+=0 \tag{30}$$

where

$$B_+=e^{\psi}\left\{\left(1-\frac{2m}{r}\right)\left[\frac{1}{4}(\psi')^2+\frac{1}{2}\psi''\right]-\frac{2m'}{r^2}-\frac{m''}{r}+\frac{1}{r^2}\right.$$
$$\left.+\frac{3}{2r^2}(r-m)\psi'-\frac{1}{r^2}(\lambda^2+\mu_0^2r^2)\right\}$$
$$+\frac{i\mu_0}{\lambda-i\mu_0 r}\left[\frac{1}{r}\left(1-\frac{m}{r}\right)+\frac{1}{2}\left(1-\frac{2m}{r}\right)\psi'-\frac{m'}{r}\right]e^{\psi} \tag{31}$$

$$\psi''=\frac{\partial^2\psi}{\partial r^2},\qquad m''=\frac{\partial^2 m}{\partial r^2}$$

Using the generalized tortoise coordinates (5), we can write equation (30) as

$$\frac{[2\varkappa(r-r_H)+1](r-2m)e^{\psi}-2r\dot{r}_H}{2\varkappa r(r-r_H)}\frac{\partial^2 R_+}{\partial r_*^2}+2\frac{\partial^2 R_+}{\partial r_*\,\partial v_*}$$
$$+\left\{\frac{2r\dot{r}_H-(r-2m)e^{\psi}}{r(r-r_H)[2\varkappa(r-r_H)+1]}+\left(1-\frac{2m}{r}\right)\psi' e^{\psi}-\frac{2(m'r-m)}{r^2}e^{\psi}\right.$$
$$+\frac{3r-5m}{r^2}e^{\psi}-\frac{m'}{r}e^{\psi}+\frac{i\mu_0}{\lambda-i\mu_0 r}\left(1-\frac{2m}{r}\right)e^{\psi}-\frac{\dot{r}_H}{2\varkappa(r-r_H)+1}$$
$$\left.\times\left(\frac{\psi}{r}-\psi'+\frac{i2\mu_0}{\lambda-i\mu_0 r}\right)\right\}\frac{\partial R_+}{\partial r_*}+\frac{2\varkappa(r-r_H)}{2\varkappa(r-r_H)+1}\left[\left(\frac{\psi}{r}-\psi'+\frac{i2\mu_0}{\lambda-i\mu_0 r}\right)\right.$$
$$\left.\times\frac{\partial R_+}{\partial v_*}+B_+R_+\right)\Bigg]=0 \tag{32}$$

When r goes to $r_H(v_0)$ and v goes to v_0, the equation can be reduced to

$$\frac{\partial^2 R_+}{\partial r_*^2}+2\frac{\partial^2 R_+}{\partial v_*\,\partial r_*}+E\frac{\partial R_+}{\partial r_*}=0 \tag{33}$$

where

$$E \equiv \frac{1}{r_H}\left(1 - \frac{M}{r_H}\right) e^{\psi} + \dot{r}_H \psi' - \frac{m'}{r_H} e^{\psi} \tag{34}$$

Here, we have used equations (9) and (12). The ingoing wave solution and the outgoing wave solution are, respectively,

$$R_+^{\text{in}} = e^{-i\omega v_*} \tag{35}$$

$$R_+^{\text{out}} = e^{-i\omega v_*} e^{2i\omega r_*} e^{-Er_*} \qquad (r > r_H) \tag{36}$$

The outgoing wave can be rewritten as

$$R_+^{\text{out}} = e^{-i\omega v_*} e^{2i\omega r}(r - r_H)^{i\omega/\varkappa}\, e^{-Er_*} \tag{37}$$

It is not analytical at the horizon. But we can continue the outgoing wave from outside of the black hole into its inside, so we have

$$R'_{\text{out}} = e^{-i\omega v_*} e^{i2\omega r_*} e^{-Er_*} e^{\pi\omega/\varkappa} e^{i\pi E/2\varkappa} \qquad (r < r_H) \tag{38}$$

The relative scattering probability produced by the horizon is

$$\left|\frac{R_{\text{out}}}{R'_{\text{out}}}\right|^2 = e^{-2\pi\omega/\varkappa} \tag{39}$$

Then we obtain the spectrum of the Hawking radiation of the Dirac particles from the black hole (Sannan, 1988)

$$N_\omega = [\exp(\omega/k_B T) + 1]^{-1} \tag{40}$$

where T is given by equation (21).

In equations (9) and (21), both the location and the temperature of the event horizon of the black hole are shown. In equations (20) and (40), we give the Hawking thermal radiation spectra of the Klein–Gordon particles and the Dirac particles, respectively. When the radiation is very weak, $\dot{r}_H$ (hence $\dot{m}$) will be very small. If we only consider the first approximation of $\dot{m}$, equations (9) and (21) reduce to those obtained by Balbinot and Barletta (1989),

$$r_H = 2M(1 + 4\dot{M}) \tag{41}$$

$$T = \frac{1}{2\pi k_B} \frac{1 - 2m' - 4\dot{M}}{4M} \tag{42}$$

So, our method is valid, and is simpler and more exact than the old method for calculating the vacuum expectation value of the renormalized energy-momentum tensor.

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